JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 16, No. 6, November–December 1993

H_∞ Norm Sensitivity Formula with Control System Design Applications

Daniel P. Giesy
Lockheed Engineering and Sciences Company, Hampton, Virginia 23666
and
Kyong B. Lim
NASA Langley Research Center, Hampton, Virginia 23681

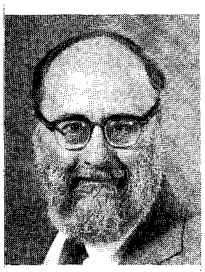
An analytic formula for the sensitivity of singular value peak variation with respect to parameter variation is derived. As a corollary, the derivative of the H_∞ norm of a stable transfer function with respect to a parameter is presented. It depends on the first derivative of the transfer function with respect to the parameter. If the transfer function has a linear system realization whose matrices depend on the parameter, then an analytic formula for this derivative is derived, and an efficient algorithm for calculating the H_∞ norm sensitivity is described. Examples are given that provide numerical verification of the H_∞ norm sensitivity formula and that demonstrate its utility in designing control systems satisfying H_∞ norm constraints.

I. Introduction

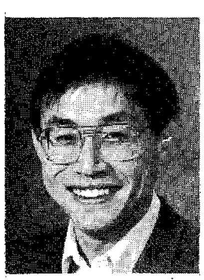
FOR the past decade, the H_∞ norm has been and currently is widely used in the analysis and design of linear feedback control systems. A selection of recent papers that use the H_∞ norm for robust control is given in Ref. 1. The reason for its popularity has been mainly the need to address multivariable control systems whose characteristics can be conveniently described by frequency-dependent singular values of various transfer function matrices (Ref. 2, Chap. 2 and Ref. 3). Physically, the H_∞ norm can be interpreted as the worst case or the largest value of the function matrix and is naturally suited to describe the worst case disturbance, uncertainty, or a performance specification for a linear system.

Currently, the standard H_∞ control problem is well understood (see, for example, Ref. 2, Chap. 3), and a reliable numerical algorithm is available to obtain minimum H_∞ norm control.[4] However, no method has yet been reported in the open literature that would optimize H_∞ norm with respect to plant parameters or a combination of plant parameters and controller gains. In this paper, we present an analytical formula for the sensitivity of the H_∞ norm so that H_∞ norms can be optimized directly via nonlinear programming with respect to arbitrary parameters including plant parameters if necessary. It should be noted that, as in any sensitivity analysis, the sensitivity of the H_∞ norm by itself does not guarantee anything. However, it can be a useful tool for the analyst or the designer to efficiently influence or improve judiciously chosen H_∞ norms whose values directly quantify various system properties of interest such as stability robustness or nominal performance. Obviously, the main payoff in using an analytical formula is that it eliminates the need to differentiate the H_∞ norm numerically, which means that the norm need not be computed so accurately, resulting in computational savings. This savings is particularly true since the H_∞ norm computation is an iterative process.

In Sec. II, we determine an analytic formula for the sensitivity of a singular value peak to parameter variation. The sensitivity of the H_∞ norm to parameter variation is found by



Daniel Giesy received his B.A. and M.A. from Ohio State University in 1960 and his Ph.D. in 1964 from the University of Wisconsin, Madison, majoring in mathematics. He then worked on the mathematics faculties at the University of Southern California, Western Michigan University, and Norfolk State University. Since 1977 he has provided technical support services under contract to NASA Langley Research Center. He has designed algorithms and written software for control system design and analysis, specializing in optimization and numerical linear algebra. He is now associated with the Analysis and Design Methods Team of the Controls/Structures Integration program.



Kyong Been Lim received the Ph.D. degree in engineering science and mechanics from Virginia Tech in 1986 and is currently a Research Engineer at NASA Langley Research Center in Hampton, Virginia, in the Spacecraft Controls Branch of Guidance and Control Division. His current areas of interest include analytical dynamics, robust control, and actuator and sensor placement of flexible space structures. He is a Member of AIAA.

Received June 8, 1992; revision received Oct. 20, 1992; accepted for publication Nov. 18, 1992.

applying this formula to the dominant peak. However, the information for other peaks can be useful, for example, in the context of optimization.

For example, if the H_∞ norm of a transfer function is the objective or an active constraint in an optimization procedure, then the optimization itself will tend to reduce the dominant peak of the singular value curves. Other peaks, not being influenced by the optimization procedure, can be expected to increase. Ultimately, two or more peaks will be codominant. At such a point, the H_∞ norm will have a derivative discontinuity. Numerically, this discontinuity can express itself as bad derivative information going into the optimization procedure's calculation of a search direction, resulting in a bad search direction. This can lead to premature indications of convergence.

One ad hoc fix for this problem is to track peaks of the singular value curves individually and use each peak value as a constraint, or, in the case that minimal H_∞ norm is the objective, using these several peak values in a multi-objective minimization scheme. For this, we need knowledge of sensitivity of each peak height to parameter variation.

To clarify the phrase "sensitivity of a singular value peak to parameter variation," we let G be a complex matrix-valued function of two real variables x and p. We consider the singular values of G as a function of x and determine the sensitivities of the peak values (local extrema) with respect to variation of p. Although this is related to the sensitivity of singular values to parameter variation, formulas for which are known (see Ref. 5, Theorem 1), it is complicated by the need to take into account the sensitivity with respect to parameter variation of the point at which the peak occurs. Dealing with this complication occupies a substantial fraction of Sec. II. Ultimately, these sensitivities depend on some singular vectors of $G(x,p)$ and the first derivative of G with respect to p.

In control system design, a common object, whose H_∞ norm is considered, is the transfer function of a linear system, which is realized by matrices that depend on one or more design parameters. So in Sec. III, if

$$G(j\omega,p)=C(p)\left[j\omega I-A(p)\right]^{-1}B(p)+D(p)$$

a formula for the first partial derivative of G with respect to p is determined in terms of the derivatives of A, B, C, and D with respect to p. Suggestions are given on how to modify existing software to use this to efficiently calculate singular value peak sensitivity.

In Sec. IV we present two examples that illustrate the significance of the H_∞ norm and its sensitivity in the context of two familiar linear feedback control systems. The first example is concerned with a numerical verification of the sensitivity formula through a comparison with sensitivities based on numerical differencing. The physical significance of the H_∞ norm whose sensitivity is computed is self-evident for the single-loop control system. The second example is concerned with a demonstration of the usefulness of the analytical formula in the context of a stability robustness optimization problem. It is shown that through a redesign of an initial controller that makes the closed-loop system initially unstable (much less robust), robust controllers that yield stable closed-loop systems can be obtained by a simple numerical optimization procedure. The iterations for the redesign will clearly indicate that the optimization amounts to a tradeoff between nominal performance and robustness of the feedback control system.

Notation. Denote the real (respectively, complex) numbers by $\boldsymbol{R}$ (respectively, $\boldsymbol{C}$), so that, for example, real vectors of length n are in $\boldsymbol{R}^n$ and complex k by n matrices are in $\boldsymbol{C}^{k\times n}$.

Definition 1. By a *real domain* we mean a nonempty connected open subset of the real numbers; i.e., a nonempty open interval, one or both of whose endpoints may be infinite.

II. Derivation of $\partial\|G\|_\infty/\partial p$

Definition 2 (SVD): Let G be a $k\times m$ matrix of real or complex numbers. Denote $\min(k,m)$ by l. Then a singular value decomposition (SVD) of G is a factorization

$$G=U\Sigma V^H \tag{1}$$

where $\Sigma=\mathrm{diag}(\sigma_1,\ldots,\sigma_l)$ is a real $k\times m$ diagonal matrix with $\sigma_1\geq\sigma_2\geq\cdots\geq\sigma_l\geq 0$, U and V are unitary matrices ($U^HU=I$ and $V^HV=I$ where H denotes conjugate transpose and I is the identity matrix) of dimensions $k\times k$ and $m\times m$, respectively, which are real if G is real and complex if G is complex. The diagonal elements of Σ are called the singular values of G. A singular value σ_i is simple if for $j\neq i$ then also $\sigma_j\neq\sigma_i$. Column i of U (respectively, V) is called the left (respectively, right) singular vector of G corresponding to σ_i.

The SVD always exists. The proof given in Ref. 6, Theorem 2.5.1 for the real case extends directly to the complex, or see Ref. 7 for an eigenvalue-based proof.

Definition 3: Assume $G:X\times P\to\boldsymbol{C}^{k\times m}$ is a real analytic complex matrix valued function of two real variables where X and P are real domains. For each fixed $p\in P$ we consider the singular values of $G(x,p)$ as a function of x. By a *simple singular value peak* of G for p, we mean a point $(x,\sigma(G(x,p)))$ where σ is a simple singular value of $G(x,p)$ and the function s_p, defined by $s_p(x)=\sigma(G(x,p))$, has a local maximum or nonzero local minimum at x.

We will derive a formula for the sensitivity of σ to variations in p where, as p varies, x also varies to track the peak.

Definition 4: Let $G:S\subseteq\boldsymbol{C}\to\boldsymbol{C}^{k\times m}$ be a matrix valued function of a complex variable whose domain S includes the imaginary axis. Then the H_∞ norm of G is denoted by $\|G\|_\infty$ and defined to be

$$\|G\|_\infty=\sup_{\omega\in R}\sigma_1(G(j\omega)) \tag{2}$$

where σ_1 denotes the maximum singular value as in Definition 2.

Now assume $G:S\times P\to\boldsymbol{C}^{k\times m}$ where S is as above and P is a real domain. As a corollary of the peak sensitivity formula, we derive a formula for $\partial\|G(\cdot,p)\|_\infty/\partial p$, the derivative (sensitivity) of this norm with respect to variation of a parameter. Some fairly general conditions will be given under which this formula holds.

To begin with, we recall the following result:

Theorem 1. Let $p_0\in\boldsymbol{R}^n$ and let G be a complex matrix valued function that is real analytic in a neighborhood of p_0. Let σ be a nonzero simple singular value of $G(p_0)$. Then there is a function $\sigma(\cdot)$ that is real analytic on a neighborhood $\mathcal{P}$ of p_0 such that for $p\in\mathcal{P}$, $\sigma(p)$ is a simple singular value of $G(p)$.

Proof. This result follows by applying Ref. 5, Lemma 1 to the case of a simple singular value, or see Ref. 8, Theorem 1.1. □

This allows us to track singular value peaks analytically.

Definition 5: Let X and P be real domains, and let $G:X\times P\to\boldsymbol{C}^{k\times m}$. If $\mathcal{P}\subset P$, $\mathcal{P}$ is a real domain, and $f:\mathcal{P}\to X$, then f is called a *singular value peak tracking function* if for each $p\in\mathcal{P}$, $(f(p),\sigma(G(f(p),p)))$ is a simple singular value peak of G for p.

Theorem 2. Let X, P, and G be as in Definition 5, and suppose G is real analytic. Suppose that for fixed $p_0\in P$ and $x_0\in X$, $(x_0,\sigma(G(x_0,p_0)))$ is a simple singular value peak of $G(x_0,p_0)$ for p_0. Suppose

$$\frac{\partial^2\sigma(G(x_0,p_0))}{\partial x^2}\neq 0 \tag{3}$$

Then there exists a unique singular value peak tracking function f on some neighborhood $\mathcal{P}$ of p_0 such that $f(p_0)=x_0$, and f is real analytic. In addition

$$\frac{\mathrm{d}f(p_0)}{\mathrm{d}p}=-\frac{\dfrac{\partial^2\sigma(G(x_0,p_0))}{\partial p\,\partial x}}{\dfrac{\partial^2\sigma(G(x_0,p_0))}{\partial x^2}} \tag{4}$$

Proof. We will apply the Implicit Function Theorem to show that the function f is defined implicitly by solving the equation

$$\frac{\partial \sigma(G(x,p))}{\partial x} = 0 \tag{5}$$

for x and imposing the condition that $f(p_0)=x_0$. Since G is real analytic, Theorem 1 provides the necessary partial derivatives. The "simple singular value peak" hypothesis gives us the boundary condition. Since we have Eq. (3), the Implicit Function Theorem applies to show that f exists, is unique, and is differentiable; and its derivative is given by Eq. (4). Real analyticity follows since $\mathrm{d}f/\mathrm{d}x$ is a ratio of real analytic functions. Since f satisfies Eq. (5), $\partial\sigma(G)/\partial x$ is zero at $(f(p), p)$. It follows from Eq. (3) that, for p in some neighborhood of p_0

$$\left.\frac{\partial^2 \sigma(G(x,y))}{\partial x^2}\right|_{(x,y)=(f(p),p)}$$

keeps the same sign, so $s_p : x \to \sigma(G(x,p))$ has a local extremum at $x=f(p)$ of the same sense that s_{p_0} has at $x_0=f(p_0)$. □

We can now use the function f of the previous theorem to track a simple singular value peak of G as we vary p. As p varies, so does the singular value peak height $\sigma(G(f(p), p))$, and it is the sensitivity of this last quantity to variation in p that we wish to determine.

Theorem 3. Let X and P be real domains and let $G : X \times P \to \boldsymbol{C}^{k\times m}$ be real analytic. Suppose $p_0 \in P$, $\mathcal{P} \subset P$ is a neighborhood of p_0, and $f : \mathcal{P} \to X$ is a real analytic singular value peak tracking function for one of the singular values σ of G. Define H by

$$H(p) = \sigma(G(f(p), p)) \tag{6}$$

Denote $f(p_0)$ by x_0. Let u and v be, respectively, left and right singular vectors of $G(x_0,p_0)$ for the singular value $\sigma(G(x_0, p_0))$. Then

$$\frac{\mathrm{d}H(p_0)}{\mathrm{d}p} = \Re\left[u^H\left(\left.\frac{\mathrm{d}G(f(p),p)}{\mathrm{d}p}\right|_{p=p_0}\right)v\right] \tag{7}$$

where $\Re$ denotes the real part of a complex number. Furthermore,

$$\left.\frac{\mathrm{d}G(f(p),p)}{\mathrm{d}p}\right|_{p=p_0} = \frac{\partial G(x_0,p_0)}{\partial x}\frac{\mathrm{d}f(p_0)}{\mathrm{d}p} + \frac{\partial G(x_0,p_0)}{\partial p} \tag{8}$$

Remark: $\mathrm{d}H(p_0)/\mathrm{d}p$ is the sensitivity of the simple singular value peak $\sigma(G(x_0, p_0))$ to variation in p.

Proof. Equation (7) follows from Ref. 5, Theorem 1.2. Equation (8) is an application of the chain rule. □

As a corollary to this, we obtain a singular value peak sensitivity formula.

Corollary 1. Let X and P be real domains and let $G : X \times P \to \boldsymbol{C}^{k\times m}$ be real analytic. Suppose $p_0 \in P$ and $x_0 \in X$ are such that $(x_0, \sigma(G(x_0, p_0)))$ is a simple singular value peak for G and that $\partial^2\sigma(G(x_0, p_0))/\partial x^2 \neq 0$. Suppose that u and v are the left and right singular vectors of $\sigma(G(x_0, p_0))$. Then the sensitivity of $\sigma(G(x, p))$ to variation in p at p_0 as x tracks the singular value peak that passes through $(x_0, \sigma(G(x_0, p_0)))$ is

$$\Re\left(u^H\left(\frac{\mathrm{d}G(x_0,p_0)}{\mathrm{d}p}\right)v\right) \tag{9}$$

Proof. By Theorem 2, there exists a peak tracking function f that interpolates (x_0,p_0). Theorem 3 then applies. Define H as in Theorem 3 by Eq. (6). Then the desired sensitivity is given by Eq. (7). Substituting Eq. (8) into Eq. (7) and multiplying out, we get

$$\frac{\mathrm{d}H(p_0)}{\mathrm{d}p} = \Re\left[\left(u^H\frac{\partial G(x_0,p_0)}{\partial x}v\right)\frac{\mathrm{d}f(p_0)}{\mathrm{d}p} + \left(u^H\frac{\partial G(x_0,p_0)}{\partial p}v\right)\right]$$

By Ref. 5, Theorem 1.2

$$\left(u^H\frac{\partial G(x_0,p_0)}{\partial x}v\right) = \frac{\partial\sigma(G(x_0,p_0))}{\partial x}$$

But, since as a function of x, $\sigma(G(x, p_0))$ has a peak (local extremum) at x_0, $\partial\sigma(G(x_0, p_0))/\partial x = 0$, and it follows that Eq. (9) is the desired sensitivity. □

As a special case of Corollary 1, we now obtain the analytic formula for the sensitivity of the H_∞ norm to parameter variation. Unfortunately, it is necessary to state the corollary with some awkward hypotheses to be sure to include some very important cases.

The trouble we are trying to avoid is the following: given p_0, if ω_1 and ω_2 are distinct real numbers for which

$$\|G(\cdot, p_0)\|_\infty = \sigma_1(G(j\omega_1, p_0)) = \sigma_1(G(j\omega_2, p_0))$$

and if the sensitivities of the singular value peaks of G at the two w to variation of p are *different*, then the H_∞ norm of G has a derivative discontinuity at p_0, so the hypotheses of Corollary 2 must exclude this case. However, in what may be the most interesting case for applying this result, the H_∞ norm is achieved at two distinct values of ω. This is the case that $G(s,p)$ is a complex-analytic function of the complex variable s and is real-valued for values of s on the real axis. In this case, $G(-j\omega,p)$ is the complex conjugate of $G(j\omega,p)$, and so has the same singular values. This does not harm the differentiability of the H_∞ norm since, in this case, the sensitivities of singular value peaks to p-variation at $\pm\omega$ are also the same. To accommodate this case, one can choose $E \subset [0,\infty)$ in Corollary 2.

Another phenomenon that could cause a derivative discontinuity in the H_∞ norm is, if for some finite ω_1 and some choice of the $\pm$ sign

$$\|G(\cdot,p_0)\|_\infty = \sigma_1(G(j\omega_1,p_0)) = \lim_{\omega\to\pm\infty}\sigma_1(G(j\omega,p_0))$$

To avoid this, we assume that G rolls off below its H_∞ norm at infinite frequency uniformly over p in some neighborhood of p_0. With this assumption, we may assume that E is compact, so the "sup" in the definition of H_∞ norm becomes "max."

Corollary 2. Let $S \subset \boldsymbol{C}$ contain the imaginary axis $j\boldsymbol{R}$ and let P be a real domain. Let $G : S \times P \to \boldsymbol{C}^{k\times m}$ be real-analytic on $j\boldsymbol{R} \times P$. Assume that there exists a compact $E \subset \boldsymbol{R}$ such that for all $p \in P$

$$\|G(\cdot,p)\|_\infty = \max_{\omega\in E}\sigma_1(G(j\omega,p))$$

Let $\omega(p) = \{\omega \in E : \|G(\cdot,p)\|_\infty = \sigma_1(G(j\omega,p))\}$. Fix $p_0 \in P$. Assume that there is a neighborhood $\mathcal{P}$ of p_0 such that for $p \in \mathcal{P}$, $\omega(p)$ contains exactly one element. Denote the single element of $\omega(p_0)$ by ω_0 and assume that ω_0 is an interior point of E, that σ_1 is a simple singular value of $G(j\omega_0,p_0)$, and that

$$\partial^2\sigma_1(G(j\omega_0,p_0))/\partial\omega^2 \neq 0$$

Then $(\omega_0, \sigma_1(G(j\omega_0, p_0)))$ is a simple singular value peak of $G(j\cdot,\cdot)$ for p_0 and $\mathrm{d}\|G(\cdot,p_0)\|_\infty/\mathrm{d}p$ exists and is equal to the sensitivity of the singular value peak at (ω_0, p_0).

Proof. $\sigma_1(G(\omega, p_0))$ has a global maximum at $\omega=\omega_0$, and σ_1 is assumed to be a simple singular value. Therefore, $(\omega_0, \sigma_1(G(j\omega_0, p_0)))$ is a simple singular value peak of $G(j\cdot,\cdot)$

for p_0. Theorem 2 applies to show that there is a unique peak-tracking function f interpolating (ω_0, p_0). It follows that, in their common domain, $\omega(p) = \{f(p)\}$. Since $\|G(j\cdot, p)\|_\infty = \sigma_1(G(f(p), p))$, the remainder of this corollary follows from Corollary 1. □

III. Derivatives of a Linear System Transfer Function Matrix

As mentioned in Sec. I, contemporary multivariable control system design techniques are often interested in the H_∞ norm of the frequency response matrix of a linear system. This directs our interest to the behavior of the transfer function matrix on the imaginary axis. If the matrices that constitute a realization of the system depend on a parameter, this takes the form

$$G(j\omega, p) = C(p)\left[j\omega I - A(p)\right]^{-1}B(p) + D(p) \tag{10}$$

The H_∞ norm sensitivity of G was expressed in Corollary 2 in terms of the singular value peak sensitivity of G, which was, in turn, expressed in Corollary 1 in terms of the singular vectors of the maximum singular value and $\partial G/\partial p$. In this section, we develop an expression for this partial derivative of the G given in Eq. (10) and indicate how to modify existing software for the efficient evaluation of G to also evaluate the singular value peak sensitivity. We first recall some elementary differentiation formulas for matrices.

Theorem 4. Let A and B be differentiable (i.e., elementwise differentiable) matrices of a real or complex variable p and denote $\mathrm{d}A/\mathrm{d}p$ and $\mathrm{d}B/\mathrm{d}p$ by A' and B'. Then for $n > 0$

$$\frac{\mathrm{d}(AB)}{\mathrm{d}p} = A'B + AB' \tag{11}$$

$$\frac{\mathrm{d}(A^n)}{\mathrm{d}p} = A'A^{n-1} + AA'A^{n-2} + \cdots + A^{n-1}A' \tag{12}$$

and

$$\frac{\mathrm{d}(A^{-n})}{\mathrm{d}p} = -A^{-n}A'A^{-1} - A^{-n+1}A'A^{-2} - \cdots - A^{-1}A'A^{-n} \tag{13}$$

Remark: Recall that, in general, matrix multiplication does not commute. This observation applies to the product of a matrix and its derivative. Therefore, the order of factors must be strictly observed in these formulas. This is why the formulas for the derivatives of matrix powers are not nearly as nice as the corresponding scalar formulas.

Proof. Equation (11) is proved by differentiating individual entries of AB. Equation (12) follows from Eq. (11) by induction on n. Equation (13) is derived by applying Eq. (11) to the identity

$$(A^{-n})(A^n) \equiv I$$

substituting Eq. (12) in the result, and solving for $\mathrm{d}(A^{-n})/\mathrm{d}p$. □

Theorem 5. Let G be given by Eq. (10). Suppose that A, B, C, and D are differentiable functions of p. Denote the derivatives by A', etc. Then

$$\frac{\partial G}{\partial p} = C'(j\omega I - A)^{-1}B + C(j\omega I - A)^{-1}A'(j\omega I - A)^{-1}B + C(j\omega I - A)^{-1}B' + D' \tag{14}$$

Proof. One easily determines that $\partial(j\omega I - A)/\partial p = -A'$. The rest follows by application of Theorem 4. □

Remark: Other derivatives such as $\partial G/\partial\omega$ and $\partial^2 G/\partial\omega^2$, which are potentially useful in locating simple singular value peaks and, in particular, in calculating the H_∞ norm, follow similarly from the observation that $\partial(j\omega I - A)/\partial\omega = jI$.

Laub[9] provides an efficient way to calculate G as given by Eq. (10) for a fixed value of p and multiple values of ω. We outline the Laub algorithm here and indicate the changes to it that will also improve the efficiency of the calculation of the sensitivity of a simple singular value peak. In the following, we understand A, B, C, and D to be evaluated at a fixed value of p, and suppress reference to it.

Laub's idea is to find an equivalent representation of the linear system transfer function matrix for which the computation is more efficient. Specifically, given A, B, and C, Laub shows how to find $\hat{A}$, $\hat{B}$, and $\hat{C}$ such that $\hat{A}$ is lower Hessenberg (i.e., zero below the first subdiagonal) and

$$\hat{C}(sI - \hat{A})^{-1}\hat{B} \equiv C(sI - A)^{-1}B$$

Both of these expressions have the form $SH^{-1}T$. The calculation of this commonly starts by factoring H into triangular factors (actually, one of the factors may incorporate pivoting information, but this does not materially affect the amount of calculation done). If H is $n \times n$ and fully populated, this takes $\mathcal{O}(n^3)$ operations. However, if H is Hessenberg, Laub provides software that can do this in $\mathcal{O}(n^2)$ operations. Applying this to $(sI - \hat{A})^{-1}$ as s varies provides the efficiency of Laub's algorithm.

A typical application involving H_∞ norm sensitivity might start by determining the H_∞ norm of $G(\cdot, p)$ for fixed p, which is now a vector of parameters. This determination could require evaluation of $G(j\omega, p)$ for multiple values of ω, so that the Laub transformation of the A, B, and C matrices would already have been done. Furthermore, knowledge of the first two ω derivatives of the maximum singular value of $G(j\omega, p)$ could be useful in finding the H_∞ norm. This in turn requires the first two ω derivatives of G (see the Remark after Theorem 5). To make use of calculations already done by the Laub software, it must be modified to return the triangular factorization of $(j\omega I - \hat{A})$ to the user.

Then, to calculate the derivative of G used in the H_∞ norm sensitivity formula, the transformation T used by Laub to convert A, B, and C into $\hat{A}$, $\hat{B}$, and $\hat{C}$ must be retained for application to the matrices $\mathrm{d}A/\mathrm{d}p_i$, etc. Also, given the triangular factorization of a Hessenberg H, the Laub software includes a subroutine to calculate $H^{-1}b$ for a column vector b. This subroutine is descended from a Linpack routine that can also calculate cH^{-1} for a row vector c. This capability should be restored to the Laub version.

The calculation now proceeds using the formula for H_∞ norm sensitivity that results from substituting Eq. (14) into Eq. (9) and multiplying out. Each term has the form $u^H M v$, where M is a matrix product of factors such as $\hat{C}$, $\hat{B}$, $(j\omega I - \hat{A})^{-1}$, A', B', C', D', and T and T^{-1}, where $\hat{A} = T^{-1}AT$. By calculating from the ends toward the middle, all operations are matrix-vector operations, which are more efficient than matrix-matrix operations. The multiplication by A', B', C', or D' is saved to last. That way, if there are several parameters for which the H_∞ norm sensitivity is to be computed, all of the other calculations are done once and reused with each new set $A' = \partial A/\partial p_i$, etc., of system sensitivity matrices. Additional efficiencies are obtained by identifying common subexpressions, judicious switching between real and complex arithmetic, and, if $\partial G/\partial\omega$ has been computed previously, use of intermediate results from that calculation.

IV. Applications to Control System Design

In this section, two numerical examples are given. The first example is concerned with validating the analytical formula for the H_∞ norm derivative by comparing it to finite-difference derivatives. The H_∞ norm, whose derivatives are computed, is motivated by a typical single-input, single-output (SISO) control design problem. The second example is given to demonstrate how the sensitivity formula could be used to improve a property of a linear feedback system by redesigning the controller. The improvement is concerned with stability robustness with respect to unstructured additive uncertainties of a typical flexible structure.

A. Validation of Analytical Formula

To provide numerical evidence of the validity of our analytic formula for H_∞ norm sensitivity, we use a SISO system consisting of a simple damped oscillator with a proportional-integral (PI) controller (see Fig. 1). The transfer function matrix T from (r,v) to (e,w) captures various physical properties of the feedback system (see, for example, Ref. 10, Chap. 3). Its H_∞ norm measures the effects of plant variation and command input on control effort and tracking error, so a designer might well want sensitivity information to help in adjusting this.

The state equations describing this feedback system are given as follows:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\xi} \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -(1+k_P)\omega_o^2 & -2\zeta_o\omega_o & \omega_o^2 \\ -k_I & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \xi \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ \omega_o^2 k_P & -\omega_o^2 k_P \\ k_I & -k_I \end{bmatrix} \begin{pmatrix} r \\ v \end{pmatrix}$$

$$\begin{pmatrix} e \\ w \end{pmatrix} = \begin{bmatrix} -W & 0 & 0 \\ -k_P & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \xi \end{pmatrix} + \begin{bmatrix} W & -W \\ k_P & -k_P \end{bmatrix} \begin{pmatrix} r \\ v \end{pmatrix}$$

The parameters appearing previously are given by

$$(\omega_o \quad \xi_o \quad W \quad k_P \quad k_I) = (5 \quad 0.01 \quad 1.0 \quad 0.01 \quad 0.05) \qquad (15)$$

The transfer function matrix from (r,v) to (e,w) is written as $T(s,k_P,k_I)$, where s denotes the Laplace variable, and k_P and k_I are the design parameters. Note that the plant parameters ω_o and ξ_o could have been included.

A central difference formula is used with forward and backward differences of 10^{-6}, and the results are shown in Table 1. The close agreement between finite-difference and analytically computed values lends credibility to the analytical formula derived herein.

B. Optimization for Stability Robustness

In this example, the analytical sensitivity formula derived herein is applied to a control design problem involving disturbance attenuation of a large flexible structure using a linear quadratic Gaussian (LQG) controller.[11] A critical requirement

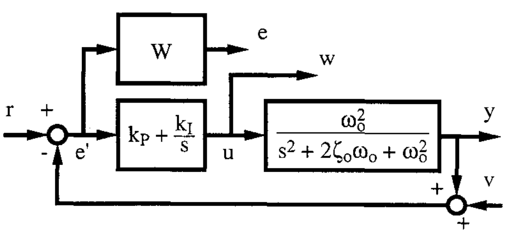


Fig. 1 Block diagram for SISO example.

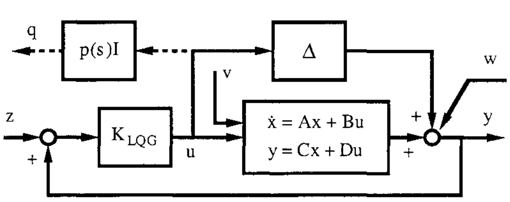


Fig. 2 Block diagram for MIMO example.

Table 1 Comparison of H_∞ norm sensitivity

Sensitivity	$\frac{\partial \lVert T(j\omega_0,k_P,k_I)\rVert_\infty}{\partial k_P}$	$\frac{\partial \lVert T(j\omega_0,k_P,k_I)\rVert_\infty}{\partial k_I}$
Finite difference	67.397077713	59.351268872
Analytical	67.397077786	59.351268894

in the application of LQG control to a typical large flexible structure is stability robustness. The robustness problem is inevitable for various reasons, which include errors caused by 1) finite discretization of an infinite dimensional problem, 2) model reduction for computational and real-time implementability, 3) modeling inaccuracies due to inaccurate physical assumptions including nonlinearities, and 4) limitations on system identification of the real system in the laboratory. Among the various uncertainties that exist in the feedback control system, we elect to investigate the stability robustness with respect to the truncated higher frequency modes, i.e., the spillover problem.

Figure 2 shows the block diagram of the feedback control system. The plant G consists of a reduced-order model of a finite element model of the Control-Structures Interaction Evolutionary Model (CEM). The details of the CEM are given in Refs. 12 and 13. The control design model consists of 22 states with 8 acceleration outputs and 8 thruster inputs. The 22 states correspond to the lowest 11 structural modes, and the remaining modes are truncated.[13] The objective of the control is to suppress vibration of the very lightly damped CEM structure using an LQG controller.

Given the regulator weights (Q,R) and estimator weights (V,W), an LQG controller is obtained. As is typical, the initial LQG controller may give suitable regulation performance when the reduced-order model is used in the evaluation. However, an eigenvalue evaluation based on a full-order model that consists of 25 structural modes showed closed-loop instability (see Fig. 5, initial design). This unacceptable instability of the closed-loop system using the initial LQG controller is not unexpected since the uncertain higher frequency dynamics were not considered in the design. This is true even if the LQG design is based on a full-order model that is assumed to be known exactly, since uncertain higher frequency modes other than the full-order modes will always exist for a large flexible structure.[14] In addition, the full-order model is generally not well known, or, at best, difficult to obtain from laboratory tests.

In this example we improve the stability robustness of the controller with respect to the truncated uncertain high frequency dynamics by introducing an unstructured additive uncertainty Δ.[15] Nonlinear programming is used to optimize the stability robustness expressed as an H_∞ norm condition, i.e., the LQG controller is robustly stable with respect to unstructured additive model uncertainties if and only if

$$\lVert \Delta K(I - GK)^{-1} \rVert_\infty < 1 \qquad (16)$$

For a detailed description of the use of the previous condition for the CEM structure, we refer the interested reader to Ref. 13. Our analytical formula for the H_∞ norm sensitivity is used in the optimization. The design variables chosen for the robustness optimization are the diagonal and upper subdiagonal elements of the Cholesky factor matrix L of the input covariance matrix so that

$$V = LL^T \geq 0 \qquad (17)$$

The number of design variables selected is 43. Figure 3 shows the design strategy for updating the LQG controller to satisfy the H_∞ norm robustness condition. The optimizer used is a minimum norm correction or, equivalently, a steepest descent approach. After each minimum norm correction, the estimator Riccati equation is resolved and the H_∞ norm recomputed. For each iteration, a single evaluation of the H_∞ norm and its

Table 2 Open-loop eigenvalues

Re(λ)	Im(λ)	ω	ζ
−4.6216e−03	±9.2431e−01	9.2432e−01	5.0000e−03
−4.6827e−03	±9.3654e−01	9.3655e−01	5.0000e−03
−4.8761e−03	±9.7521e−01	9.7522e−01	5.0000e−03
−2.2936e−02	±4.5871e+00	4.5872e+00	5.0000e−03
−2.3493e−02	±4.6985e+00	4.6986e+00	5.0000e−03
−2.7457e−02	±5.4913e+00	5.4914e+00	5.0000e−03
−4.6290e−02	±9.2580e+00	9.2581e+00	5.0000e−03
−5.4605e−02	±1.0921e+01	1.0921e+01	5.0000e−03
−5.9155e−02	±1.1831e+01	1.1831e+01	5.0000e−03
−7.2300e−02	±1.4460e+01	1.4460e+01	5.0000e−03
−8.9175e−02	±1.7835e+01	1.7835e+01	5.0000e−03
−1.2612e−01	±2.5225e+01	2.5225e+01	5.0000e−03
−1.2667e−01	±2.5335e+01	2.5335e+01	5.0000e−03
−1.3212e−01	±2.6425e+01	2.6425e+01	5.0000e−03
−1.3797e−01	±2.7595e+01	2.7595e+01	5.0000e−03
−1.7283e−01	±3.4566e+01	3.4566e+01	5.0000e−03
−1.9414e−01	±3.8827e+01	3.8827e+01	5.0000e−03
−1.9574e−01	±3.9149e+01	3.9149e+01	5.0000e−03
−2.0329e−01	±4.0657e+01	4.0658e+01	5.0000e−03
−2.0954e−01	±4.1907e+01	4.1908e+01	5.0000e−03
−2.3161e−01	±4.6321e+01	4.6322e+01	5.0000e−03
−2.6054e−01	±5.2107e+01	5.2108e+01	5.0000e−03
−2.8168e−01	±5.6336e+01	5.6337e+01	5.0000e−03
−3.9225e−01	±7.8450e+01	7.8451e+01	5.0000e−03
−5.2945e−01	±1.0589e+02	1.0589e+02	5.0000e−03

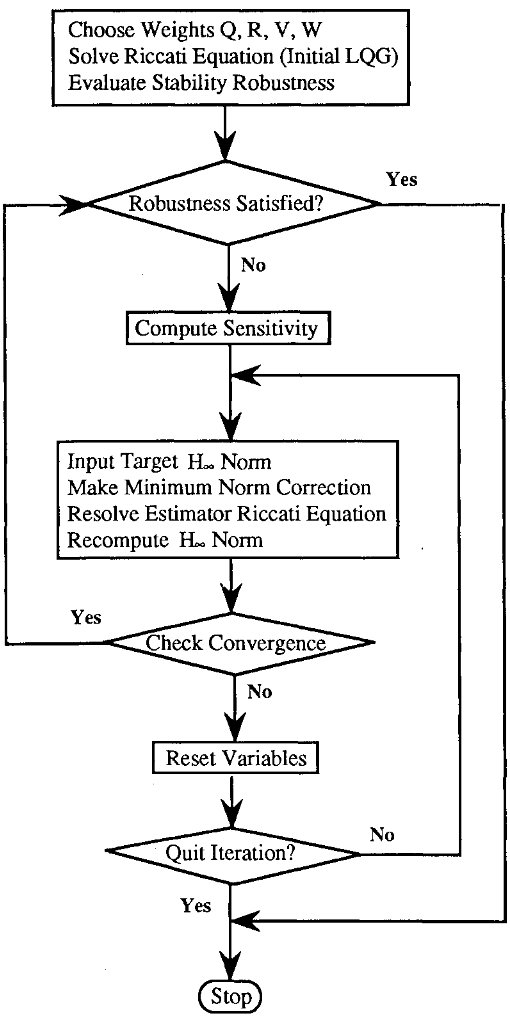


Fig. 3 Design strategy for LQG controller with H_∞ norm constraint.

Table 3 Closed-loop eigenvalues without H_∞ norm constraint

Re(λ)	Im(λ)	ω	ζ
−1.1930e−01	±9.1974e−01	9.2744e−01	1.2863e−01
−6.4839e−01	±9.6003e−01	1.1585e+00	5.5969e−01
−1.2558e+00	±6.7220e−01	1.4244e+00	8.8164e−01
−7.7242e−01	±1.2000e+00	1.4271e+00	5.4124e−01
−1.3739e+00	±6.8455e−01	1.5350e+00	8.9505e−01
−1.1019e+00	±1.2059e+00	1.6335e+00	6.7456e−01
−2.6344e+00	0.0000e+00	2.6344e+00	1.0000e+00
−3.0510e−01	±4.6510e+00	4.6610e+00	6.5457e−02
−6.2578e−01	±4.6364e+00	4.6785e+00	1.3376e−01
−7.2403e−01	±4.6413e+00	4.6974e+00	1.5413e−01
−2.8417e+00	±4.5628e+00	5.3754e+00	5.2865e−01
−4.9835e−01	±5.4741e+00	5.4967e+00	9.0663e−02
−1.4255e−01	±9.2591e+00	9.2602e+00	1.5394e−02
−1.0921e+01	0.0000e+00	1.0921e+01	1.0000e+00
−4.0229e+00	±1.0233e+01	1.0995e+01	3.6589e−01
−1.3703e+00	±1.0927e+01	1.1012e+01	1.2443e−01
−8.5700e−01	±1.1849e+01	1.1880e+01	7.2136e−02
−1.2685e+00	±1.4249e+01	1.4305e+01	8.8677e−02
−1.9012e−01	±1.4463e+01	1.4465e+01	1.3144e−02
−1.7578e+01	0.0000e+00	1.7578e+01	1.0000e+00
−1.8337e+00	±1.7483e+01	1.7579e+01	1.0431e−01
−2.5707e−01	±1.7845e+01	1.7847e+01	1.4404e−02
−1.4586e+01	±1.1288e+01	1.8444e+01	7.9085e−01
−1.2590e−01	±2.5233e+01	2.5233e+01	4.9894e−03
2.7544e−02	±2.5527e+01	2.5527e+01	−1.0790e−03
−1.0519e−01	±2.6448e+01	2.6449e+01	3.9770e−03
−1.2215e−01	±2.8339e+01	2.8339e+01	4.3102e−03
−1.3186e−01	±3.4632e+01	3.4633e+01	3.8073e−03
−1.1679e−01	±3.8885e+01	3.8885e+01	3.0033e−03
3.3700e−02	±3.9457e+01	3.9457e+01	−8.5409e−04
−6.0473e−02	±4.1175e+01	4.1175e+01	1.4687e−03
2.6758e−01	±4.2036e+01	4.2037e+01	−6.3654e−03
−4.4991e+01	0.0000e+00	4.4991e+01	1.0000e+00
4.1038e−02	±4.6447e+01	4.6447e+01	−8.8354e−04
8.8084e−01	±5.2581e+01	5.2588e+01	−1.6750e−02
2.1153e+00	±5.6433e+01	5.6473e+01	−3.7457e−02
−2.8841e−01	±7.8471e+01	7.8471e+01	3.6754e−03
−3.7200e−01	±1.0592e+02	1.0592e+02	3.5119e−03

analytical sensitivities are computed. Although convergence to a global minimum is not guaranteed, the goal attainment parameter can be chosen (iteratively if necessary) to control convergence. As a rule-of-thumb in the selection of target H_∞ norms, larger steps can be taken if in the past few steps, linear corrections gave accurate predictions.

The iteration histories for the weighted transfer function are shown in Fig. 4. The optimization required 94 iterations to converge to a near-local minimum. This is evidenced by small values for the gradients. Figure 4 shows the maximum singular value plot only at a few intermediate stages. The initial LQG controller corresponds to an H_∞ norm of 55.6, whereas the optimized LQG gave a value of 1.33, i.e., almost satisfied the original requirement of 1. As expected, the maximum singular value plot is considerably flattened out with the minimization iterations. Obviously, better controllers are expected if more design variables are used.

The iteration histories for the real component of the closed-loop eigenvalues for the full-order model (50 states) are shown in Fig. 5. It can be seen that initially, a few higher frequency modes are unstable, but they are stabilized as the iteration progresses. On the other hand, several lower frequency modes lose their initially large stability margins but remain stable. Tables 2–4 show the open-loop eigenvalues for the full-order model, the closed-loop eigenvalues of the full-order model with initial LQG controller, and the closed-loop eigenvalues of the model with the optimized LQG controller, respectively. The initial open-loop system is all stable with a low damping ratio of 0.5% (see Table 2). The initial LQG very well damps the first 11 modes to which it is designed, but 6 high frequency modes are destabilized (see Table 3). The optimized LQG controller avoids the high frequency spillover, but the damping of the first 11 modes is reduced as compared to the initial LQG

Table 4 Closed-loop eigenvalues with H_∞ norm constraint

Re(λ)	Im(λ)	ω	ζ
−1.1930e−01	±9.1973e−01	9.2744e−01	1.2864e−01
−4.8426e−03	±9.3654e−01	9.3655e−01	5.1706e−03
−4.8761e−03	±9.7521e−01	9.7522e−01	5.0000e−03
−1.0109e+00	0.0000e+00	1.0109e+00	1.0000e+00
−1.1036e+00	0.0000e+00	1.1036e+00	1.0000e+00
−7.6998e−01	±1.1962e+00	1.4225e+00	5.4127e−01
−1.3918e+00	±6.7883e−01	1.5485e+00	8.9879e−01
−2.0915e+00	0.0000e+00	2.0915e+00	1.0000e+00
−2.2981e−02	±4.5871e+00	4.5872e+00	5.0098e−03
−3.0657e−01	±4.6481e+00	4.6582e+00	6.5813e−02
−1.0800e+00	±4.5759e+00	4.7016e+00	2.2971e−01
−7.2617e−01	±4.6541e+00	4.7104e+00	1.5416e−01
−6.8980e−01	±5.4063e+00	5.4502e+00	1.2656e−01
−4.9618e−01	±5.4784e+00	5.5008e+00	9.0201e−02
−1.4252e−01	±9.2586e+00	9.2597e+00	1.5391e−02
−9.1377e−02	±1.0921e+01	1.0921e+01	8.3667e−03
−1.3696e+00	±1.0919e+01	1.1004e+01	1.2446e−01
−1.6526e−01	±1.1830e+01	1.1831e+01	1.3969e−02
−8.5733e−01	±1.1840e+01	1.1871e+01	7.2220e−02
−7.2300e−02	±1.4460e+01	1.4460e+01	5.0000e−03
−1.9017e−01	±1.4459e+01	1.4461e+01	1.3151e−02
−8.9175e−02	±1.7835e+01	1.7835e+01	5.0000e−03
−2.5491e−01	±1.7834e+01	1.7836e+01	1.4292e−02
−1.2680e−01	±2.5225e+01	2.5226e+01	5.0267e−03
−1.4376e−01	±2.5344e+01	2.5345e+01	5.6722e−03
−1.3352e−01	±2.6426e+01	2.6426e+01	5.0526e−03
−1.0215e−01	±2.7616e+01	2.7617e+01	3.6990e−03
−1.7660e−01	±3.4575e+01	3.4576e+01	5.1076e−03
−1.8200e−01	±3.8839e+01	3.8839e+01	4.6860e−03
−2.0632e−01	±3.9188e+01	3.9189e+01	5.2648e−03
−1.9716e−01	±4.0703e+01	4.0703e+01	4.8439e−03
−1.0710e−01	±4.1938e+01	4.1939e+01	2.5537e−03
−1.9929e−01	±4.6352e+01	4.6352e+01	4.2994e−03
−2.6684e−01	±5.2110e+01	5.2110e+01	5.1207e−03
−2.8811e−01	±5.6341e+01	5.6342e+01	5.1135e−03
−7.0947e+01	0.0000e+00	7.0947e+01	1.0000e+00
−3.7327e−01	±7.8456e+01	7.8457e+01	4.7577e−03
−5.2160e−01	±1.0589e+02	1.0589e+02	4.9256e−03

controller (see Table 4). Assuming that the lower frequency modes are the modes to be controlled, these results clearly demonstrate the tradeoffs between nominal performance and stability robustness.

Figure 6 shows the maximum singular value $\bar{\sigma}(\Delta)$ of the additive uncertainty Δ from the truncated high frequency modes; the corresponding upper-bound low-order polynomial $p(s)$; and the initial and final stability robustness of the LQG controllers, $1/\bar{\sigma}(K(I-GK)^{-1})$. The upper-bound polynomial is chosen as

$$p(s) = \left(\frac{40}{3}\right)^3 \left(\frac{45}{500}\right) \left(\frac{s+3}{s+40}\right)^3 \left(\frac{s+500}{s+45}\right) \tag{18}$$

which satisfies $|p(j\omega)| \geq \bar{\sigma}(\Delta(j\omega))$. The polynomial is chosen to have a small positive gain at zero frequency instead of rolling off like the uncertainty $\Delta(s)$. This choice is typically necessary in real applications where sensor bias and uncertainty in the feedforward term exist for accelerometers.[13] This added robustness may lead to a slightly lower performance at low frequencies. It can be seen that, for the initial LQG controller, the stability robustness condition is severely violated, whereas the optimized LQG controller satisfies the robustness condition over nearly all frequencies.

The maximum singular values for the compensator and its loop gain matrix are shown in Figs. 7a and 7b. The control bandwidth is significantly smaller for the LQG controller with improved robustness. The controller activity is therefore significantly reduced with increasing frequency. The expected drop in performance at lower frequencies is clearly indicated by the decrease in the damping of the structural modes, some of which were close to the uncontrolled modal response.

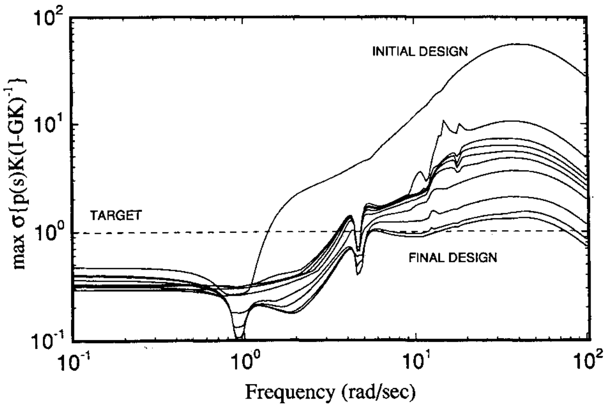


Fig. 4 Iteration history for the weighted robustness constraint.

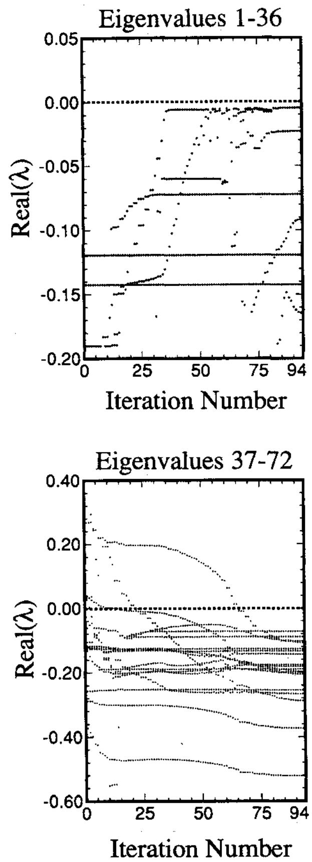


Fig. 5 Iteration histories of real components of full-order closed-loop eigenvalues.

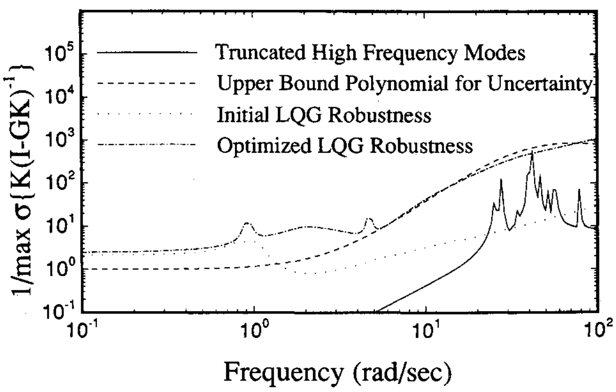


Fig. 6 Model uncertainty and stability robustness.

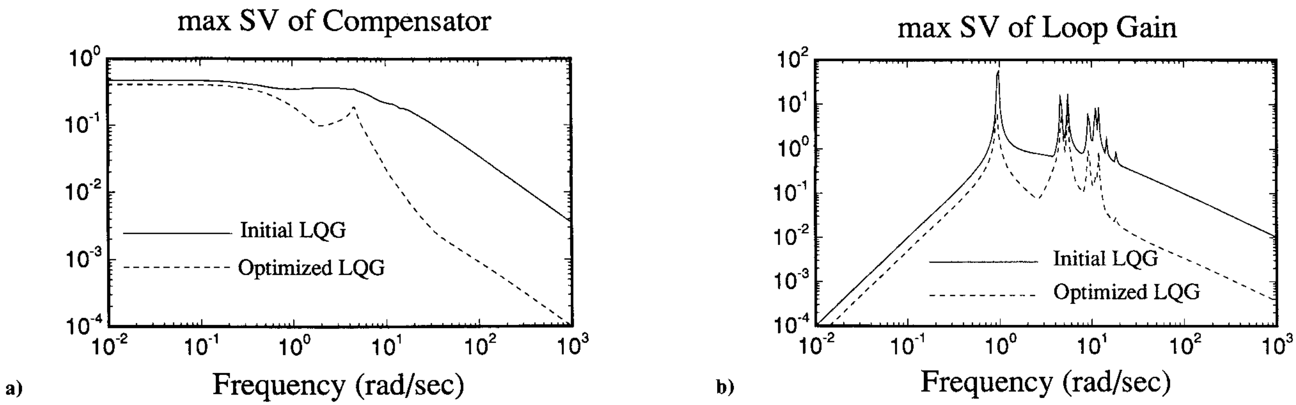


Fig. 7 Frequency response: a) LQG compensator and b) loop gain.

V. Conclusions

An analytic formula was derived for the sensitivity of singular value peaks to parameter variation. Conditions were given under which this formula may be used to find the sensitivity of the H_∞ norm to parameter variation. In the event that this function was the transfer function of a linear system whose matrices depend on a parameter, an expression for a derivative of the transfer function needed in the sensitivity formula was derived in terms of the linear transfer function matrices and their derivatives. An algorithm for efficient calculation of singular value peak sensitivities was outlined. The sensitivity formula was validated by comparing its results to sensitivities estimated by numerical differencing. A control design example was given to show how the sensitivity formula can be used to improve the stability robustness of a feedback control system quantified as an H_∞ norm.

Acknowledgment

The first author's work was supported by NASA Langley Research Center under Contract NAS1-19000.

References

[1]Dorato, P., and Yedavalli, R. K., *Recent Advances in Robust Control,* IEEE Press, New York, 1990.

[2]Francis, B. A., *A Short Course in H_∞ Control Theory,* Springer-Verlag, New York, 1987.

[3]Maciejowski, J. M., *Multivariable Feedback Design,* Addison-Wesley, Wokingham, England, UK, 1989.

[4]Doyle, J. C., Glover, K., Khargonekar, P., and Francis, B. A., "State-Space Solutions to Standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Control Problems," *IEEE Transactions on Automatic Control,* Vol. 34, No. 8, 1989, pp. 831–847.

[5]Freudenberg, J. S., Looze, D. P., and Cruz, J. B., "Robustness Analysis Using Singular Value Sensitivities," *International Journal of Control,* Vol. 35, No. 1, 1982, pp. 95–116.

[6]Golub, G. H., and Van Loan, C. F., *Matrix Computations,* 2nd ed., Johns Hopkins Press, Baltimore, MD, 1989.

[7]Forsythe, G. E., and Moler, C. B., *Computer Solution of Linear Algebraic Systems,* Prentice-Hall, Englewood Cliffs, NJ, 1967.

[8]Sun, J., "A Note on Simple Non-Zero Singular Values," *Journal of Computational Mathematics,* Vol. 6, July 1988, pp. 258–266.

[9]Laub, A. J., "ALGORITHM 640: Efficient Calculation of Frequency Response Matrices from State Space Models," *ACM Transactions on Mathematical Software,* Vol. 12, March 1986, pp. 26–33.

[10]Franklin, G. F., Powell, J. D., and Emami-Naeini, A., *Feedback Control of Dynamic Systems,* Addison-Wesley, Reading, MA, 1986.

[11]Kwakernaak, H., and Sivan, R., *Linear Optimal Control Systems,* Wiley, New York, 1972, Chap. 5.

[12]Belvin, W. K., et al., "Langley's CSI Evolutionary Model: Phase 0," NASA TM 104165, Nov. 1991.

[13]Lim, K. B., Maghami, P. G., and Joshi, S. M., "A Comparison of Controller Designs for an Experimental Flexible Structure," *IEEE Control Systems,* Vol. 12, June 1992, pp. 108–118.

[14]Balas, M. J., "Trends in Large Space Structure Control Theory: Fondest Hopes, Wildest Dreams," *IEEE Transactions on Automatic Control,* Vol. AC-27, 1982, pp. 522–535.

[15]Doyle, J. C., and Stein, G., "Multivariable Feedback Design: Concepts for a Classical/Modern Synthesis," *IEEE Transactions on Automatic Control,* Vol. AC-26, 1981, pp. 4–16.